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Dimers on a Simple-Quartic Net with a Vacancy

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A seminal milestone in lattice statistics is the exact solution of the enumeration of dimers on a simple-quartic net obtained by Fisher, Kasteleyn, and Temperley (FKT) in 1961. An outstanding related and yet unsolved problem is the enumeration of dimers on a net with vacant sites. Here we consider this vacant-site problem with a single vacancy occurring at certain specific sites on the boundary of a simple-quartic net. First, using a bijection between dimer and spanning tree configurations due to Temperley, Kenyon, Propp, and Wilson, we establish that the dimer generating function is independent of the location of the vacancy, and deduce a closed-form expression for the generating function. We next carry out finite-size analyses of this solution as well as that of the FKT solution. Our analyses lead to a logarithmic correction term in the large-size expansion for the vacancy problem with free boundary conditions. A concrete example exhibiting this difference is given. We also find the central charge $c=-2$ in the language of conformal field theory for the vacancy problem, as versus the value $c=1$ when there is no vacancy.

KEY WORDS: Dimers; spanning trees; finite-size analysis.

1. INTRODUCTION

The problem of enumerating close-packed dimers on a finite simple-quartic net was solved by Temperley and Fisher[(1, 2)] and by Kasteleyn[(3)] in 1961. An outstanding related but yet unsolved problem is the enumeration of dimers on a net with vacant sites.[(4)] Here, we consider this vacancy problem when a single vacant site occurs on the boundary.

The difficulty associated with the vacancy problem is that, while the determinant whose square root yields the dimer generating function can be

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written down using the Kasteleyn formulation,[3] its evaluation is difficult. In 1974 Temperley[5] reported an intriguing bijection relating close-packed dimer coverings with spanning tree configurations on two related lattices. This offers an alternate approach to the vacancy problem since spanning trees can be enumerated by standard means. The Temperley bijection has been of renewed recent interest and extended to graphs with certain weighted and/or directional edges.[6] Here, we use an extension of the Temperley bijection due to Kenyon *et al.*[6] to study the vacancy dimer problem.

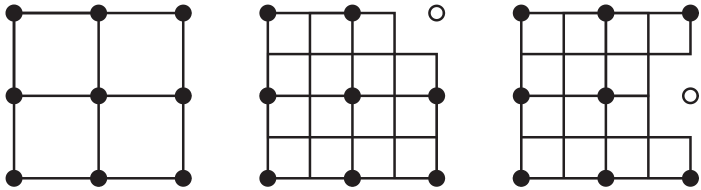
Our first result is that, for a simple-quartic net with free boundaries and one fixed vacant site located at certain specific sites on the boundary, the dimer generating function is independent of the position of the vacancy. The exact generating function for close-packed dimers on this net is then deduced from that of the spanning trees. In view of the connection with the conformal field theory[7] and current interests in finite-size analyses for two-dimensional lattice models,[8–10] we next carry out finite-size analyses for the weighted spanning tree solution as well as that of the FKT solution. It is found that a logarithmic correction term arises in the large-size expansion in the case of the vacancy problem with free boundaries, a term which is absent in the expansion of the FKT solution. A concrete example demonstrating this difference is given. We also find that the occurrence of a vacancy yields a new central charge $c = -2$ in the language of the conformal field theory.

The organization of this paper is as follows: To make the paper self-contained, we restate and establish in Section 2 the bijection due to Temperley[5] between spanning tree and dimer configurations, as well as an extended version of the bijection due to Kenyon *et al.*[6] This extended Temperley bijection permits us to establish in Section 3 the independence of the dimer generating function on the location of the vacancy when it occurs at specific boundary sites. The explicit expression of the generating function for a simple-quartic net is then obtained. In Section 4 we carry out finite-size analyses for weighted spanning trees and the FKT dimer solution. Specializing the results to dimer enumerations, we find a logarithmic correction term which is unique to the vacancy problem with free boundaries. Discussions and a summary of our findings are given in Section 5.

2. THE EXTENDED TEMPERLEY BIJECTION

For definiteness we consider a simple-quartic net (lattice) with free boundaries, although much of the results of this section also hold for more general planar graphs.[4,6]

First we restate the Temperley bijection. Starting from a $L_1 \times L_2$ simple-quartic net G with free boundaries, one constructs a dimer lattice G_D



(a) (b) (c)

Fig. 1. The construction of dimer lattices G_D from a 3×3 spanning tree lattice G. Solid circles denote odd sites and open circles denote the odd site that has been removed in G_D. (a) A spanning tree lattice G. (b) A dimer lattice G_D constructed from G with one corner site removed. (c) A dimer lattice G_D constructed from G with one odd site on the boundary removed.

by (i) adding a new site at the midpoint of each edge of G, (ii) inserting in each internal face of G a new site connected to the midpoints of the 4 edges of G surrounding it, and (iii) removing one corner site of the resulting lattice and its incident edges on G_D. Thus, G_D has a total of $(2L_1-1)(2L_2-1)-1$ sites consisting of the original L_1L_2-1 sites of G, which we call the *odd* sites, and the remaining $(2L_1-1)(2L_2-1)-L_1L_2$ new sites, which we call the *even* sites. An example of this construction for $L_1 = L_2 = 3$ is shown in Figs. 1(a) and 1(b).

A spanning tree is a collection of connected edges of G which does not form closed circuits and covers all sites. Then we have the

Temperley bijection: There exists a one–one correspondence between spanning tree configurations on G and dimer configurations on G_D.

To see that the bijection holds, one observes that to each spanning tree configuration on G, one can construct a unique dimer configuration on G_D by first laying a dimer along each tree edge, starting from the edge(s) covering the corner site of G_D which has (have) been removed, and proceed along the spanning tree edges in an obvious fashion. After laying dimers along all tree edges, the remaining sites of G_D can then be covered by dimers in a unique way.[5] Conversely, starting from each dimer configuration on G_D, one constructs a unique tree configuration on G by drawing bonds (tree edges) along dimers originating from all odd sites. These bonds cannot form close circuits, since otherwise they would have enclosed an odd number of sites of G_D which is not permitted in close-packed dimer configurations. This process leads to a unique tree configuration on G. This completes the proof. An example of the Temperley bijection is shown in Figs. 2(a) and 2(b).

Kenyon *et al.*[6] have shown that the Temperley bijection holds more generally for graphs with certain weighted and/or directed edges. For our

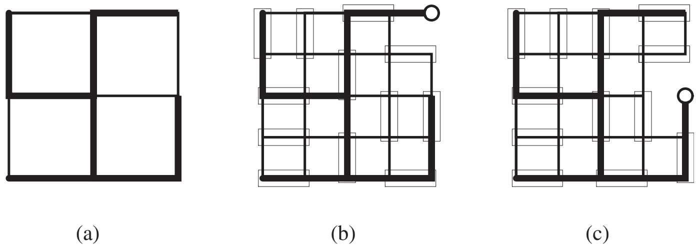


Fig. 2. The bijection between spanning trees on G and dimer configurations of G_D. (a) A spanning tree configuration on G. (b) The corresponding dimer configuration on the dimer lattice G_D of Fig. 1(b). (c) The corresponding dimer configuration on the dimer lattice G_D of Fig. 1(c).

purposes, however, we shall confine ourselves to the original Temperley bijection as stated in the above, with the step iii) replaced by one of removing an odd site on the boundary together with its incident edges. An example of constructing such a G_D is shown in Fig. 1(c). The proof of the bijection between tree configurations on G and dimer coverings on G_D goes through as before, and we are led to the

Extended Temperley bijection (Temperley–Kenyon–Propp–Wilson): There exists a one–one correspondence between spanning trees on G and dimer coverings on any G_D constructed from G by removing any boundary odd site and its incident edges in step (iii) of the construction described in the above.

An example of such a bijection is shown in Figs. 2(a) and 2(c).

Remark. The extended Temperley bijection does not hold for dimer lattices G_D containing an interior vacancy. In that case while each spanning tree can still be mapped into a unique dimer configuration as before, there exist dimer configurations which cannot be mapped into spanning trees. These are dimer coverings with no dimers laying on any of the 4 edges of G incident to the defect site.

3. DIMER LATTICE WITH A VACANT BOUNDARY SITE

3.1. Dimer Generating Function

The dimer generating function for G_D is

$$Z(G_D; x_1, x_2) = \sum_{\text{dimer config.}} x_1^{n_1} x_2^{n_2} \tag{1}$$

where the summation is taken over all dimer covering configurations, x_1 and x_2 are, respectively, the weights of horizontal and vertical dimers, and n_1 and n_2 are, respectively, the number of horizontal and vertical dimers. Clearly, we have

$$Z(G_D; 1, 1) = \text{the number of dimer configurations on } G_D. \tag{2}$$

Consider two different dimer lattices G_D and G'_D obtained from G as described in the above, namely, by removing different boundary odd sites. Then we have the following equivalence:

Proposition 3.1.1.

$$Z(G_D; x_1, x_2) = Z(G'_D; x_1, x_2) \tag{3}$$

for any G_D and G'_D.

Proof. The extended Temperley bijection dictates that there is a one–one correspondence between spanning tree configurations on G and dimer configurations on *any* G_D. It follows that there is a bijection between dimer coverings on G_D and G'_D, and that the summation in (1) on G_D can be considered as taken over all spanning tree configurations on G.

For each spanning tree configuration T of G, the dimer weight in the summand in (1) consists of two factors,

$$x_1^{n_1} x_2^{n_2} = W_o(T; x_1, x_2)\, W_e(T; x_1, x_2), \tag{4}$$

where W_o is the product of the weights of those dimers originating from odd sites, and W_e is the product of the weights of those dimers covering two even sites. For the two dimer coverings of G_D and G'_D corresponding to the same T, their W_e factors are the same by definition. Their W_o factors are also the same since, even though the respective dimer positions may be shifted, they lay along the same spanning tree edges hence carry the same weights. It follows that summations on the l.h.s. and r.h.s. of (3) are identical term by term, and the proposition is proved. ∎

Remark. Proposition 3.1.1 holds more generally for arbitrary planar G and its related G_D. Since the overall dimer weight can always be factorized into the product W_oW_e as in (4), the proof of the proposition goes through as presented.

We next consider the generating function of *weighted* spanning trees for the $L_1 \times L_2$ simple-quartic G. Assign weights x_1 and x_2, respectively, to edges in the horizontal and vertical direction. Then, the weighted spanning tree generating function is

$$T(G; x_1, x_2) = \sum_T x_1^{n_1} x_2^{n_2} \tag{5}$$

where the summation is taken over all spanning tree configurations T on G and, as in (1), n_1 and n_2 are the numbers of edges in the spanning tree in the respective directions. Particularly, we have

$$T(G; 1, 1) = \text{the number of spanning tree configurations on } G. \tag{6}$$

From the extended Temperley bijection, it is clear that we can also write (5) as

$$T(G; x_1, x_2) = \sum_T W_o(T; x_1, x_2) \tag{7}$$

where W_o is the factor in (4) for the dimer covering on any G_D. It is seen from (7) that if all dimers covering even sites of G_D have the weight 1, namely $W_e = 1$, then the dimer generating function is simply $T(G; x_1, x_2)$.

More generally for a simple-quartic G of size $L_1 \times L_2$ and the related G_D, we have the equivalence:

Proposition 3.1.2.

$$Z(G_D; x_1, x_2) = x_1^{L_1(L_2-1)} x_2^{L_2(L_1-1)} T\left(G; \frac{x_1}{x_2}, \frac{x_2}{x_1}\right). \tag{8}$$

Proof. From the construction of G_D we note that there are a total of $L_2(L_1 - 1)$ even sites located at midpoints of horizontal edges of G. We call these the H sites. Similarly, there are $L_1(L_2 - 1)$ even sites of G_D located at midpoints of vertical edges of G, which we call the V sites.

The H and V sites can be covered by either horizontal or vertical dimers. Let N_{Hh} (N_{Hv}) be the number of horizontal (vertical) dimers covering the H sites. Then we have

$$N_{Hh} + N_{Hv} = L_2(L_1 - 1). \tag{9}$$

Likewise, we have

$$N_{Vh} + N_{Vv} = L_1(L_2 - 1), \tag{10}$$

where N_{Vh} (N_{Vv}) is the number of horizontal (vertical) dimers covering the V sites. In these notation, we can rewrite the spanning tree and dimer generating functions as

$$\begin{aligned} T(G; x, y) &= \sum_T x^{N_{Hh}} y^{N_{Vv}} \\ Z(G_D; x_1, x_2) &= \sum_T x_1^{N_{Hh}+N_{Vh}} x_2^{N_{Hv}+N_{Vv}}, \end{aligned} \tag{11}$$

where we have used the one–one correspondence between spanning tree configurations T and dimer configurations. Proposition 3.1.2 now follows after eliminating N_{Vh} and N_{Hv} in (11) using (9) and (10). ▮

3.2. Dimer Enumerations

For a simple-quartic G of size $L_1 \times L_2$ with free boundaries, the generating function (7) for weighted spanning trees has been evaluated[11] and is given by

$$T(G; x_1, x_2) = \frac{1}{L_1 L_2} \prod_{m=0}^{L_1-1} \prod_{n=0}^{L_2-1} \left[2x_1 \left(1 - \cos\frac{m\pi}{L_1}\right) + 2x_2 \left(1 - \cos\frac{n\pi}{L_2}\right) \right],$$
$$(m, n) \neq (0, 0). \tag{12}$$

Now the dimer lattice G_D is of size $M \times N$ with a boundary vacancy, where

$$M = 2L_1 - 1, \qquad N = 2L_2 - 1. \tag{13}$$

In terms of the dimer lattice sizes M, N, we thus have, after using Proposition 3.1.2 and (12) and some steps

$$\begin{aligned} Z_{\{M\times N-1\}}(x_1, x_2) &= x_1^{(M-1)/2} x_2^{(N-1)/2} \\ &\quad \times \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} \left[4x_1^2 \cos^2\frac{m\pi}{M+1} + 4x_2^2 \cos^2\frac{n\pi}{N+1} \right]. \end{aligned} \tag{14}$$

Here, we must have $MN = \text{odd}$ to admit dimer coverings. The subscript $\{M \times N - 1\}$ in (14) reminds us that the enumeration is for an $M \times N$ net with one boundary odd site removed. This expression is to be compared with the enumeration of dimers on an $M \times N$ simple-quartic net without vacancies. For M and N both even, for example, the expression is[3]

$$Z_{\{M,N\}}(x_1, x_2) = \prod_{m=1}^{M/2} \prod_{n=1}^{N/2} \left[4x_1^2 \cos^2\left(\frac{m\pi}{M+1}\right) + 4x_2^2 \cos^2\left(\frac{n\pi}{N+1}\right) \right]. \tag{15}$$

4. FINITE-SIZE ANALYSES

Finite-size expansions of physical quantities associated with two-dimensional lattice models have been of current interest both in physics[7, 9, 10] and in mathematics.[4] For the dimer problem Kenyon[4] has recently deduced very general results on the leading terms of the asymptotic expansion of the dimer enumeration for rectilinear lattices with free boundaries of any shape. Alternately, one can obtain expansions for regular lattices, in principle to all orders, by analyzing known exact expressions.[8, 9] This is the approach we now use.

4.1. Spanning Tree Generating Function

Consider first the generating function (12) for the fully weighted spanning trees. For large L_1 and L_2, we expect to have

$$\frac{1}{L_1 L_2} \ln T(G; x_1, x_2) = f_{\text{bulk}}(x_1, x_2) + f_c(x_1, x_2) \tag{16}$$

where f_{bulk} is the per-site bulk free energy and f_c is the correction containing terms of the order of L_1^{-1}, L_2^{-1} and higher. Using (12), we find the bulk free energy

$$\begin{aligned} f_{\text{bulk}}(x_1, x_2) &\equiv \lim_{L_1, L_2 \to \infty} \frac{1}{L_1 L_2} \ln T(G; x_1, x_2) \\ &= \frac{1}{\pi^2} \int_0^{\pi} d\theta \int_0^{\pi} d\phi \ln[2x_1(1-\cos\theta) + 2x_2(1-\cos\phi)] \\ &= \frac{4}{\pi} \int_0^{\pi/2} d\phi \ln(\sqrt{x_1 + x_2 \sin^2\phi} + \sqrt{x_2} \sin\phi), \end{aligned} \tag{17}$$

where the last line is obtained by carrying out the θ integration. The computation of correction terms f_c for products of the form of (12) is standard.[2, 8, 12] Particularly, one has

$$f_{\text{bulk}}(1, 1) = \frac{4}{\pi} G \tag{18}$$

where G is the Catalan constant given by

$$G = 1 - 3^{-2} + 5^{-2} - 7^{-2} + \cdots = 0.915\,965\,594\ldots. \tag{19}$$

To compute (16) we proceed as follows. Take out a factor x_1 from each of the $L_1 L_2 - 1$ factors in (12) and split the product into 3 parts to take care of the exclusion of the $m = n = 0$ factor. We have

$$T(G; x_1, x_2) = (L_1 L_2)^{-1}\, x_1^{L_1 L_2 - 1}(T_0 T_1 T_2) \tag{20}$$

where

$$T_0 = \prod_{m=1}^{L_1-1} \prod_{n=1}^{L_2-1} F(m, n), \qquad T_1 = \prod_{m=1}^{L_1-1} F(m, 0), \qquad T_2 = \prod_{n=1}^{L_2-1} F(0, n) \tag{21}$$

with

$$F(m, n) = 2\left(1 - \cos\frac{m\pi}{L_1}\right) + 2\tau\left(1 - \cos\frac{n\pi}{L_2}\right) \tag{22}$$

and

$$\tau = x_2/x_1. \tag{23}$$

Using the identity[13]

$$\prod_{n=1}^{N-1}\left[2\cosh 2\theta - 2\cos\frac{n\pi}{N}\right] = \frac{\sinh 2N\theta}{\sinh 2\theta}, \tag{24}$$

and its $\theta \to 0$ limit,

$$\prod_{n=1}^{N-1}\left[2 - 2\cos\frac{n\pi}{N}\right] = N, \tag{25}$$

we find

$$T_0 = \prod_{n=1}^{L_2-1} \frac{\sinh(2L_1\theta_n)}{\sinh 2\theta_n}, \qquad T_1 = L_1, \qquad T_2 = \tau^{L_2-1} L_2, \tag{26}$$

with θ_n given by

$$\cosh 2\theta_n = 1 + \tau\left(1 - \cos\frac{n\pi}{L_2}\right), \tag{27}$$

or, explicitly,

$$\theta_n = F\left(\frac{n\pi}{2L_2}\right) \equiv [\cosh^{-1}(1+2a_n^2)]/2$$
$$= \sinh^{-1} a_n$$
$$= \ln(a_n + \sqrt{1+a_n^2}), \tag{28}$$

where

$$a_n = \sqrt{\tau} \sin \frac{n\pi}{2L_2}. \tag{29}$$

Substituting (26) into (12), we thus obtain

$$T(G; x_1, x_2) = x_1^{L_1 L_2 - 1} \tau^{L_2 - 1} \prod_{n=1}^{L_2-1} \frac{\sinh(2L_1\theta_n)}{\sinh 2\theta_n}. \tag{30}$$

The product in the denominator in (30) can again be evaluated using (24) as

$$\prod_{n=1}^{L_2-1} \sinh^2 2\theta_n = \prod_{n=1}^{L_2-1} (\cosh 2\theta_n - 1) \cdot \prod_{n=1}^{L_2-1} (\cosh 2\theta_n + 1)$$
$$= \prod_{n=1}^{L_2-1} \left[\tau\left(1-\cos\frac{n\pi}{L_2}\right)\right] \cdot \prod_{n=1}^{L_2-1} \left[2+\tau-\tau\cos\frac{n\pi}{L_2}\right]$$
$$= L_2 \left(\frac{\tau}{2}\right)^{2(L_2-1)} \left(\frac{\sinh 2L_2\alpha}{\sinh 2\alpha}\right) \tag{31}$$

where α is given by

$$\cosh 2\alpha = 1 + 2\tau^{-1} \tag{32}$$

or $\sinh \alpha = 1/\sqrt{\tau}$, or explicitly,

$$\alpha = \ln(\sqrt{\tau^{-1}} + \sqrt{1+\tau^{-1}}). \tag{33}$$

Combining these results, we obtain from (30) the expression

$$T(G; x_1, x_2) = x_1^{L_1 L_2 - 1} \left(\frac{\sinh 2\alpha}{L_2 \sinh 2L_2\alpha}\right)^{1/2} \prod_{n=1}^{L_2-1} [2\sinh(2L_1\theta_n)]. \tag{34}$$

Taking the logarithm, we obtain

$$\ln T(G; x_1, x_2) = (L_1 L_2 - 1) \ln x_1 + 2L_1 \sum_{n=1}^{L_2-1} \theta_n + \sum_{n=1}^{L_2-1} \ln(1 - e^{-4L_1\theta_n})$$
$$- L_2\alpha - \frac{1}{2}\ln(1 - e^{-4L_2\alpha}) - \frac{1}{2}\ln L_2 + \frac{1}{2}\ln(\sinh 2\alpha). \quad (35)$$

For large L_1 and L_2 with the ratio L_1/L_2 finite, the first two terms in (35) contribute to the bulk free energy $f_{\text{bulk}}(x_1, x_2)$ given in (17). To carry out the summations in (35), we use the Euler–MacLaurin summation formula given by

$$\sum_{n=1}^{N} f(a+n\delta) = \frac{1}{\delta}\int_a^{a+N\delta} f(x)\, dx + \frac{1}{2}[f(a+N\delta) - f(a)]$$
$$+ \frac{\delta}{12}[f'(a+N\delta) - f'(a)] + O(\delta^3). \quad (36)$$

With $a = 0$, $N = L_2$, $\delta = \pi/2L_2$, and $f(x) = F(x)$ defined in (28), one has

$$\sum_{n=1}^{L_2-1} \theta_n = \sum_{n=1}^{L_2} \theta_n - \theta_{L_2}$$
$$= \frac{L_2}{2}[f_{\text{bulk}}(x_1, x_2) - \ln x_1] - \frac{1}{2}\ln(\sqrt{1+\tau} + \sqrt{\tau}) - \frac{\pi\sqrt{\tau}}{24L_2} + O(L_2^{-3}). \quad (37)$$

For the second summation in (35), we follow the manipulation in ref. 8 to write

$$\sum_{n=1}^{L_2-1} [1 - e^{-4L_1\theta_n}] = \sum_{n=1}^{\infty} (1 - e^{-2nL_1\pi\sqrt{\tau}/L_2}) + O(L_2^{-2+\epsilon}) \quad (38)$$

for some $0 < \epsilon < 2$, with $\epsilon \to 0$ when $L_2 \to \infty$. Putting the results together, we find the finite-size correction

$$f_c(x_1, x_2) = \frac{c_1(x_1, x_2)}{L_1} + \frac{c_2(x_1, x_2)}{L_2} + \frac{c_3(x_1, x_2)}{L_1 L_2} + o\left(\frac{1}{L_1 L_2}\right), \quad (39)$$

where

$$c_1(x_1, x_2) = -\ln(\sqrt{\tau^{-1}} + \sqrt{1+\tau^{-1}})$$

$$c_2(x_1, x_2) = -\ln(\sqrt{\tau} + \sqrt{1+\tau})$$

$$c_3(x_1, x_2) = -\frac{1}{2}\ln L_2 + \ln 2 + \frac{1}{4}\ln\frac{(x_1+x_2)}{x_1^3 x_2^2} - \frac{\pi\sqrt{\tau}\,L_1}{12L_2} + \sum_{n=1}^{\infty}\ln(1-e^{-2n\pi\sqrt{\tau}\,L_1/L_2}). \tag{40}$$

Particularly, for $x_1 = x_2 = 1$, the expression $f_c(1, 1)$ given by (39) reduces to the one given in refs. 4 and 14.

Despite its appearance, the expression for $c_3(x_1, x_2)$ is actually symmetric in $\{x_1, L_1\} \leftrightarrow \{x_2, L_2\}$, a fact can be seen from the identity

$$\sum_{m=1}^{\infty}\ln(1-e^{-2m\pi L_2/\sqrt{\tau}\,L_1}) - \frac{1}{2}\ln L_1 - \frac{1}{4}\ln x_2 - \frac{\pi L_2}{12\sqrt{\tau}\,L_1}$$
$$= \sum_{n=1}^{\infty}\ln(1-e^{-2n\pi\sqrt{\tau}\,L_1/L_2}) - \frac{1}{2}\ln L_2 - \frac{1}{4}\ln x_1 - \frac{\pi\sqrt{\tau}\,L_1}{12L_2}. \tag{41}$$

Introducing the Jacobi theta function

$$\vartheta_1(\phi, q) = 2q^{1/4}\sin\phi\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2\phi + q^{4n}), \tag{42}$$

and the identity[15]

$$\prod_{n=1}^{\infty}(1-q^{-2n}) = \left[\frac{\vartheta_1'(0, q)}{2q^{1/4}}\right]^{1/3} \tag{43}$$

where ϑ_1' is the derivative of ϑ_1 with respect to ϕ, then we have also

$$c_3(x_1, x_2) = -\frac{1}{2}\ln L_2 + \frac{1}{4}\ln\frac{(x_1+x_2)}{x_1^3 x_2^2} + \frac{1}{3}\ln[4\vartheta_1'(0, q)] \tag{44}$$

where

$$q = e^{-L_1\sqrt{\tau}\,\pi/L_2}. \tag{45}$$

The identity (41) follows from the Jacobi transformation[15]

$$\vartheta_1'(0, e^{-\pi v}) = v^{-3/2}\vartheta_1'(0, e^{-\pi/v}), \qquad v > 0. \tag{46}$$

Now, the theta function ϑ_1' is finite for L_1/L_2 finite,[3] it follows that the leading behavior of c_3 is $\ln L_2$ ($\sim \ln L_1$).

In conformal field theory[7] one needs to compute the limits

$$\frac{1}{L_1}\lim_{L_2\to\infty}\frac{\ln T(\mathbf{Z}_2)}{L_2} = f_{\text{bulk}} + \frac{c_1}{L_1} + \frac{\Delta_1}{L_1^2} + o(L_1^{-2}); \tag{47}$$

$$\frac{1}{L_2}\lim_{L_1\to\infty}\frac{\ln T(\mathbf{Z}_2)}{L_1} = f_{\text{bulk}} + \frac{c_2}{L_2} + \frac{\Delta_2}{L_2^2} + o(L_2^{-2}). \tag{48}$$

Using (39), we find

$$\Delta_1 = -\frac{\pi}{12\sqrt{\tau}}, \qquad \Delta_2 = -\frac{\pi\sqrt{\tau}}{12}. \tag{49}$$

This yields a central charge $c = -2$ upon taking $x_1 = x_2 = 1$ ($\tau = 1$).

4.2. Dimer Enumerations

We are now in a position to analyze the finite-size corrections of the two dimer enumerations (14) and (15). Although the expressions refer to two dimer lattices with different geometry, one for an $M \times N - 1$ lattice with a vacancy and $MN =$ odd, and one for an $M \times N$ lattice with $MN =$ even, a comparison can still be meaningful if both expansions are expressed in terms of lattice sizes M and N.

(a) Close-Packed Dimers. For close-packed dimers on an $M \times N$ net with $MN =$ even, we have carried out the analysis for the expression (15) along the lines outlined in the above, and obtained the result (which can also be extracted from discussions in refs. 8 and 9)

$$\ln Z_{\{M\times N\}}(x_1, x_2) = (MN+1)\,\bar{f}_{\text{bulk}} + N\bar{c}_1 + M\bar{c}_2 + \bar{c}_3 + o(1), \tag{50}$$

[3] This also follows from the fact that the series $\sum_{n=1}^{\infty}\ln(1-u^n) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u^{mn}/m = -\sum_{m=1}^{\infty}u^m/m(1-u^m)$ converges for all $0 \leqslant u < 1$.

where

$$\bar{f}_{\text{bulk}}(x_1, x_2) = \frac{1}{4} f_{\text{bulk}}(x_1^2, x_2^2),$$

$$\bar{c}_1(x_1, x_2) = \bar{f}_{\text{bulk}}(x_1, x_2) - \frac{1}{2} \ln(x_1 + \sqrt{x_1^2 + x_2^2})$$

$$\bar{c}_2(x_1, x_2) = \bar{f}_{\text{bulk}}(x_1, x_2) - \frac{1}{2} \ln(x_2 + \sqrt{x_1^2 + x_2^2})$$

$$\bar{c}_3(x_1, x_2) = \frac{1}{2} \ln 2 - \frac{1}{2} \ln(x_1 + \sqrt{x_1^2 + x_2^2}) - \frac{1}{2} \ln(x_2 + \sqrt{x_1^2 + x_2^2}) + \frac{1}{4} \ln(x_1^2 + x_2^2) + \frac{\pi M x_2}{24 N x_1} + \sum_{n=1}^{\infty} \ln(1 + e^{-(2n-1)\pi M x_2 / N x_1}). \tag{51}$$

Expressions of $\bar{c}_1(x_1, x_2)$ and $\bar{c}_2(x_1, x_2)$ reduce to those found by Kenyon[4] when $x_1 = x_2 = 1$. In the language of the conformal field theory,[7] the term $\pi M x_2 / 24 N x_1$ in $\bar{c}_3$ yields the central charge $c = 1$ upon taken $M = N$ and $x_1 = x_2$, the accepted value for dimer and Ising systems.

Again, the expression (51) for $\bar{c}_3$ is symmetric in $\{x_1, M\} \leftrightarrow \{x_2, N\}$, a fact can be seen from the identity

$$\frac{\pi M x_2}{24 N x_1} + \sum_{n=1}^{\infty} \ln(1 + e^{-(2n-1)\pi M x_2 / N x_1}) = \frac{\pi N x_1}{24 M x_2} + \sum_{m=1}^{\infty} \ln(1 + e^{-(2m-1)\pi N x_1 / M x_2}). \tag{52}$$

The series $\sum_{n=1}^{\infty} \ln(1 + u^{2n-1})$ converges,[4] so $\bar{c}_3$ does not diverge for large M, N.

The expression for $\bar{c}_3$ can also be written as

$$\bar{c}_3(x_1, x_2) = \frac{2}{3} \ln 2 - \frac{1}{2} \ln(x_1 + \sqrt{x_1^2 + x_2^2}) - \frac{1}{2} \ln(x_2 + \sqrt{x_1^2 + x_2^2}) + \frac{1}{4} \ln(x_1^2 + x_2^2) + \frac{1}{2} \ln \vartheta_3(0, q) - \frac{1}{6} \ln \vartheta_1'(0, q), \tag{53}$$

where $q = e^{-\pi M x_2 / N x_1}$ and the theta function ϑ_3 is given by[15]

$$\vartheta_3(\phi, q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2\phi + q^{4n-2}). \tag{54}$$

[4] This also follows from the fact that the series $\sum_{n=1}^{\infty} \ln(1 + u^{2n-1}) = -\sum_{m=1}^{\infty} (-u)^m / m(1 - u^{2m})$ converges for all $0 \leqslant u < 1$.

Then, the identity (52) is a consequence of the Jacobi transformations (41) and

$$\vartheta_3(0, e^{-\pi/v}) = v^{1/2}\vartheta_3(0, e^{-\pi v}), \qquad v > 0. \tag{55}$$

(b) Close-Packed Dimers with a Boundary Vacancy. For the simple-quartic net $M \times N - 1$ with a boundary vacancy and $MN = \text{odd}$, one uses

$$L_1 = (M+1)/2, \qquad L_2 = (N+1)/2, \tag{56}$$

and expand (39) for large M and N. After some algebra, we find

$$\begin{aligned}
\ln Z_{\{M\times N-1\}}&(x_1, x_2)\\
&= (L_1L_2 - L_1)\ln x_1 + (L_1L_2 - L_2)\ln x_2 + \ln T\left(G; \frac{x_1}{x_2}, \frac{x_2}{x_1}\right)\\
&= (L_1L_2 - L_1)\ln x_1 + (L_1L_2 - L_2)\ln x_2\\
&\quad + L_1L_2\left[f_{\text{bulk}}\left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right) + f_c\left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right)\right]\\
&= (MN+1)\,\bar{f}_{\text{bulk}} + N\bar{c}_1 + M\bar{c}_2 + \bar{c}_3' + o(1),
\end{aligned} \tag{57}$$

where $\bar{c}_1$ and $\bar{c}_2$ are given in (51), and

$$\begin{aligned}
\bar{c}_3'(x_1, x_2) = {}& \frac{3}{2}\ln 2 - \frac{1}{2}\ln\left(x_1 + \sqrt{x_1^2 + x_2^2}\right)\\
& - \frac{1}{2}\ln\left(x_2 + \sqrt{x_1^2 + x_2^2}\right) + \frac{1}{4}\ln\left(1 + \frac{x_2^2}{x_1^2}\right)\\
& - \frac{1}{2}\ln N - \frac{\pi M x_2}{12 N x_1} + \sum_{n=1}^{\infty}\ln\left(1 - e^{-2n\pi M x_2/N x_1}\right).
\end{aligned} \tag{58}$$

Comparing $\bar{c}_3'$ with $\bar{c}_3$ given by (51), we see that for large M, N the deletion of a boundary site introduces a logarithmic correction term $-\ln\sqrt{N}$ in $\bar{c}_3'$. Furthermore, upon taking $M = N$ the term $-\pi M x_2/12 N x_1$ in $\bar{c}_3'$ yields a central charge $c = -2$, which is different from that of the dimer system without vacancies.

To verify the occurrence of a logarithmic term for the vacancy problem, we consider the following example. Consider two dimer nets of $N^2 - 1$ sites each, where $N \geqslant 3$ is an odd integer so that the nets admit dimer coverings. While the two nets have different geometries, one a rectangular net of size $(N+1)\times(N-1)$ and one a square net of size $N\times N$ with one boundary odd site removed, they have the same area and perimeter. Any difference in the evaluations of (50) and (57) would occur in $\bar{c}_3$ and $\bar{c}'_3$ and higher order terms. Now from (50) and (57) we obtain

$$\begin{aligned}\ln Z_{\{(N+1)\times(N-1)\}}(1,1) &= N^2\left(\frac{G}{\pi}\right)+2N\bar{c}_1-\ln(1+\sqrt{2})+\frac{3}{4}\ln 2+\frac{\pi}{24}\\ &\quad+\sum_{n=1}^{\infty}\ln[1+e^{-(2n-1)\pi}]+o(1)\\ \ln Z_{\{N\times N-1\}}(1,1) &= (N^2+1)\left(\frac{G}{\pi}\right)+2N\bar{c}_1-\ln(1+\sqrt{2})+\frac{7}{4}\ln 2\\ &\quad-\frac{\pi}{12}-\frac{1}{2}\ln N+\sum_{n=1}^{\infty}\ln[1-e^{-2n\pi}]+o(1).\end{aligned} \tag{59}$$

Defining the ratio

$$R(N)\equiv\frac{Z_{\{(N+1)\times(N-1)\}}(1,1)}{Z_{\{N\times N-1\}}(1,1)} \tag{60}$$

and using (59), we find the large N behavior

$$R(N)\to C\sqrt{N},\qquad N\to\infty, \tag{61}$$

where

$$\begin{aligned}C=\lim_{N\to\infty}\frac{R(N)}{\sqrt{N}}&=\frac{e^{\pi/8-G/\pi}}{2}\prod_{n=1}^{\infty}\left(\frac{1+e^{-(2n-1)\pi}}{1-e^{-2n\pi}}\right).\\ &=0.578250\ldots.\end{aligned} \tag{62}$$

As a numerical check, we have computed the value of $R(N)/\sqrt{N}$ for $N=3$ to 2251 using (14) and (15). For $N=9$, for example, one has

$$\begin{aligned}Z_{\{10\times 8\}}(1,1)&=1031151241=(89)\times(11585969)\\ Z_{\{9\times 9-1\}}(1,1)&=557568000=2^{12}\times 3^2\times 5^3\times(11)^2\\ R(9)/\sqrt{9}&=0.616457\ldots.\end{aligned} \tag{63}$$

Results plotted in Fig. 3 confirm the large N limit of C given by (62) as well as the occurrence of the logarithmic correction in the vacancy

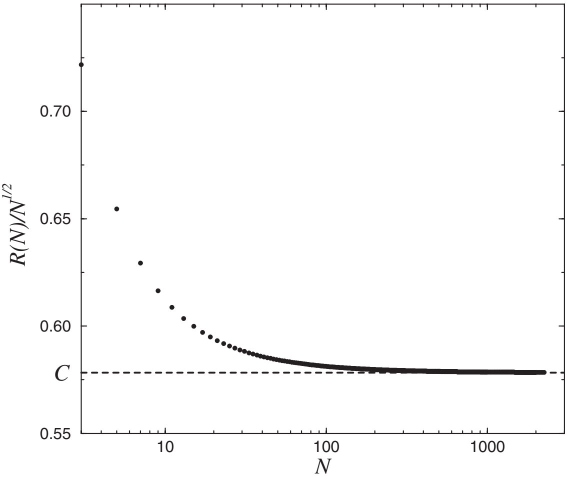


Fig. 3. The enumeration of $R(N)/\sqrt{N}$ for $N = 3$ to 2251. The dashed line indicates the value C given by (62) in the large N limit.

problem. We remark that a similar result obtained by Kenyon[4] involving the occurrence of vacancy sites in the middle of a rectilinear net gives the ratio

$$R(N) \sim N^{3/4}, \qquad N \to \infty, \tag{64}$$

and thus a logarithmic correction $-\frac{3}{4} \ln N$ in the finite-size expansion.

5. SUMMARY AND DISCUSSIONS

We have used the Temperley–Kenyon–Propp–Wilson bijection between spanning trees on a lattice with free boundaries and dimer configurations on a related lattice with a boundary vacancy to establish the independence of the dimer generating function on the location of the vacancy. The equivalence is stated in Proposition 3.1.1. The generating function for close-packed dimers on a lattice with a single boundary vacancy is next computed as given by ref. 14, and compared with that of the known results for dimers without vacancies. It is found that the vacancy introduces a logarithmic correction in the large lattice expansion. A concrete example exhibiting this correction for an $MN =$ odd net with a vacancy as compared to an $MN =$ even net without vacancies is given.

To ascertain whether the logarithmic correction is due to the defect of a vacancy, or due to the oddness of the net size, one needs to compare expansions for two nets (of the same even-even lattice), one with two boundary vacancies and one without vacancies. While this problem can be formulated as the evaluation of the inverse of a matrix[(16)] in the Kasteleyn formulation, we argue that since the correction in question is that of the physical free energy of a dimer system, on physical ground one expects the correction to be additive for vacancies located sufficiently far apart. This would imply the logarithmic correction to be a "local" property due to the occurrence of vacancies.

We have also found that in the language of the conformal field theory the central charge for the vacancy problem is $c=-2$ as compared to the value of $c=1$ for the dimer solution without vacancies. Furthermore, the $\sqrt{N}$ ratio (61) implies the existence of a boundary operator with scaling dimension $1/2$, a value which does not appear in the standard Kac classification of operators at central charge -2.[5] The extraction of the central charge should be viewed with caution, however, since the dimer systems do not exhibit critical points.

We have also carried out finite-size analyses (details of which to be given elsewhere) for spanning trees on simple-quartic nets with other, including the toroidal, cylindrical, Möbius, and Klein bottle, boundary conditions. It is found that the logarithmic correction reported in this paper arises only in the case of free boundaries. This is consistent to the fact that the formulation of the extended Temperley bijection as presented in this paper is a property that is unique to graphs with free boundaries.

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